QNo- State and Prove De Morgan's and Bertrand's Test.

Statement:- If $\sum a_n$ is a series of +ve terms such that,

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)\right\}\log n\right]=l.$$

then, $\sum a_n$ is convergent if $l>1$ and divergent when $l<1$.

Proof:- $\because \sum a_n$ is a series of +ve terms. Hence, the given series can be written as,

$$\sum a_n = a_1+a_2+a_3+\cdots+a_n+a_{n+1}+\cdots\infty \quad \text{--- (1)}$$

Let us compare this series with the auxiliary series $\sum b_n$ which is

$$\sum b_n = 1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\cdots+\frac{1}{n(\log n)^p}+\cdots \text{ to } \infty.$$

We know that this series $\sum b_n$ is Convergent when $p>1$ and divergent which $p\leq 1$.

Here, $b_n = \frac{1}{n(\log n)^p}$

$\therefore b_{n+1} = \frac{1}{(n+1)\log(n+1)^p}$

Hence, by the Comparision 2nd Test, the given series is Convergent or divergent according as.

$$\frac{a_n}{a_{n+1}} > \text{ or } < \frac{b_n}{b_{n+1}}$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \frac{(n+1)\log(n+1)^p}{n(\log n)^p}$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}} \left\{\frac{\log(n+1)}{\log n}\right\}^p$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right) \left\{\frac{\log\left(n\left(1+\frac{1}{n}\right)\right)}{\log n}\right\}^p$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right) \left\{\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right) \left\{\frac{\log n}{\log n} + \frac{\log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right) \left\{1 + \frac{\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \frac{1}{3}\left(\frac{1}{n}\right)^3 - \cdots}{\log n}\right\}^p$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right) \left\{\frac{1}{\log n} + \frac{1}{n\log n} - \frac{1}{2}\left(\frac{1}{n}\right)^2 \frac{1}{\log n} + \cdots\right\}^p$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right) \left\{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \cdots\right\}^p$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{or} < \left(1+\frac{1}{n}\right)\left\{1+\frac{p}{n\log n} - \frac{p}{2n^2\log n} + \cdots\right\}$$

from Binomial theo^m.

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{or} < 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n}-\frac{p}{2n^2\log n} + \cdots$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{or} < 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{2p-p}{2n^2\log n}+\cdots$$

$$\Rightarrow \frac{a_n}{a_{n+1}} > \text{or} < 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{2n^2\log n}+\cdots$$

$$\Rightarrow \frac{a_n}{a_{n+1}}-1 > \text{or} < \frac{1}{n}+\frac{p}{n\log n}+\frac{p}{2n^2\log n}+\cdots$$

$$\Rightarrow n\left\{\frac{a_n}{a_{n+1}}-1\right\} > \text{or} < 1+\frac{p}{\log n}+\frac{p}{2n\log n}+\cdots$$

$$\Rightarrow \left[n\left\{\frac{a_n}{a_{n+1}}-1\right\}-1\right] > \text{or} < \frac{p}{\log n}+\frac{p}{2n\log n}+\cdots$$

$$\Rightarrow \left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]\log n\Big\} > \text{or} < p+\frac{p}{2n}$$

$$\Rightarrow \lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]\log n > \text{or} < p$$

$$\Rightarrow \lim_{n\to\infty}\left[\left[n\left(\frac{a_n}{a_{n+1}}\right)-1\right]\log n\right] > \text{or} < 1.$$

In this way the theo^m is proved.